**Partition of Unity**

**Definition 1.** Let be a set. A collection of subsets of is a topology of if it satisfies the followings.



is called a ‘topological space’ and subsets of contained in are ‘open’ in *.*

**Definition 2.** Let be a topological space.

1. A set is ‘closed’ in
2. The ‘closure’ of a set is the smallest closed set in which contains .
3. A set is ‘compact’ every open covering of has a finite sub-covering.
4. A ‘neighborhood’ of a point is any open subset of which contains .
5. is a ‘Hausdorff space’ such that
6. is ‘locally compact’ such that is compact.

**Theorem 1.** Let be a topological space, is compact, and is closed in .

If , then is compact.

**Proof.** Let be an open covering of . Then implies . Since is an open covering of a compact set , there exists a finite collection such that

Hence, we get . i.e there is a finite sub-covering of for

Therefore, is compact.

**Corollary 1.** If and is compact, then is also compact.

**Theorem 2.** Let is a Hausdorff space, where is compact, and .

Then there exist open sets and such that , , and .

**Proof.** For each , there are open sets and such that , and .

Since , is an open covering of compact . Hence there are points such that

Set and . Then

**Corollary 2.**

1. Compact subsets of Hausdorff spaces are closed.
2. If is closed and is compact in a Hausdorff space, then is compact.

**Proof.** (a): For Theorem2, let and be such sets for each. Since and for each *p*, . And is open. Thus is open.

(b): It follows from Theorem 1 and (a).

**Theorem 3.** Let is a collection of compact subsets of a Hausdorff space such that .

Then there exists a finite sub-collection such that .

Proof. Put . By Corollary 2 (a), since is compact, is open for each . Since , foe each , such that . Thus, is an open covering of . Since is compact, there is a finite sub-covering of i.e . This implies that

**Theorem 4.** Suppose is open in a locally compact Hausdorff space , and is compact.

Then there is an open set such that

**Proof.** Since is locally compact, where is compact for each . Since and is compact, there are points such that covers .

Note that a finite union of sets with compact closure has a compact closure. Let . Then is open and has a compact closure. If , take .

Suppose . Theorem 2 shows that to each there corresponds an open set such that and . Hence is a collection of compact sets by Corollary 2 (b). Indeed,

By Theorem 3, there are points such that

Let . Then is open and contains , and . Since and is compact, is compact by Corollary 1. Therefore, has the required properties.

**Definition 3.** Let be a topological space and be a function.

1. is lower semi-continuous is open for every
2. is upper semi-continuous is open for every
3. Let is a function between two topological spaces and .

is continuous is open for every open subset

1. is said to be a ‘characteristic function’ of each subset of defined by

**Remark.**

1. A real valued function is continuous if and only if it is both upper and lower semi-continuous.
2. is lower semi-continuous if and only if is open.
3. is upper semi-continuous if and only if is closed.
4. is lower semi-continuous if every is lower semi-continuous.
5. is upper semi-continuous if every is upper semi-continuous.

**Definition 4.** Let be a topological space and be a function. The support of is defined by

The collection of all continuous real valued function on whose support is compact is denoted by .

Note: is a vector space over since sum of two continuous functions and a scalar multiplication of a continuous function are continuous, and if and if .

**Theorem 5.** Let is a continuous function between two topological spaces and .

If is a compact subset of , then is compact in .

**Proof.** Let be an open covering of . Then is an open covering of since is continuous. Hence for some and therefore

**Notation.**

1. is compact subset of , , , and for all .
2. is open in , , , and .
3. and

**Urysohn’s Lemma.** Suppose is a locally compact Hausdorff space, is open in , , and is compact. Then there exists a function such that

**Proof.** Put , and let be an enumeration of the rationals in . Applying Theorem 4 twice, we can choose open sets and such that they have compact closures and

Suppose and have been chosen in such a manner that implies . Then one of the numbers , say will be the largest one which is smaller than , and another, say , will be the smallest one larger than . Using Theorem 4 again, we can find so that

By mathematical induction, we obtain a collection of open sets with the following properties: , each is compact, and

Define

on for each , and

on for each .

Since and is open for each , is lower semi-continuous. Since is compact in a Hausdorff space, is closed by Corollary 1 and so is upper semi- continuous for each . Thus is lower semi-continuous and is upper semi- continuous by Remark.

Note that , that if , and that . It remains to show that and then it implies is continuous by Remark (a).

Suppose for some , and . Then , , and . It contradicts to construction of . Thus for all , so .

Suppose for some . Then there are rationals and such that . Since , we have . Since , we have . It contradicts to construction of . Hence , so is continuous.

Therefore, with ( is compact and is a closure of some set, by corollary 1), , for all , we conclude that .

**Partition of Unity for Locally Compact Hausdorff Space.**

Suppose are open subsets of a locally compact Hausdorff space , is compact, and

Then there exist functions () such that

The collection is called a ‘partition of unity’ on , subordinate to the cover .

**Proof.** By Theorem 4, for each , taking as the compact set, with compact closure for some (depending on ). Since , there are points such that . If , let

be the finite union of those which lie in . Then is compact and for each . By Urysohn’s Lemma, there are functions such that . Define

Suppose for some . Then . This means that . Since , . And implies (). Thus for each . By mathematical induction, we easily get

Since , implies that at least one contains . Since , at least one at each point . Therefore, we conclude that

**Reference.** Rudin, *Real and Complex Analysis*, 1987